

ASSOUAD DIMENSIONS OF COMPLEMENTARY SETS

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ABSTRACT. Given a positive, decreasing sequence a , whose sum is L , we consider all the closed subsets of $[0, L]$ such that the lengths of their complementary open intervals are in one to one correspondence with the sequence a . The aim of this note is to investigate the possible values that Assouad-type dimensions can attain for this class of sets. In many cases, the set of attainable values is a closed interval whose endpoints we determine.

1. INTRODUCTION

In this paper, we are interested in the Assouad dimension, introduced in [1, 2] by Assouad, and its ‘dual’, the Lower Assouad dimension, introduced in [13] by Larman. Assouad dimensions were initially of interest because of their application in the theory of embeddings of metric spaces in Euclidean spaces and in the study of quasisymmetric maps (see [12], [18] and references therein). Recently, the Assouad-type dimensions have received much attention in the fractal geometry community (c.f., [5], [7], [8], [9], [17]) as they provide quantitative information on the extreme local behaviour of the geometry of the underlying set. Indeed, if we denote by $\dim_A E$ and $\dim_L E$ the Assouad and Lower Assouad dimensions of the compact set E respectively, then the following relations hold between these dimensions and the more familiar Hausdorff, packing and upper box dimensions:

$$\dim_L E \leq \dim_H E \leq \dim_P E \leq \overline{\dim}_B E \leq \dim_A E.$$

Given $a = \{a_j\}$, a positive, decreasing sequence with finite sum L , we define the class \mathcal{C}_a to be the family of all closed subsets of $[0, L]$ whose complement (in $[0, L]$) is comprised of disjoint open intervals with lengths given by the entries of a . We call such sets the complementary sets of a . Notice that every compact subset of \mathbb{R} of zero Lebesgue measure belongs to exactly one \mathcal{C}_a and every set in \mathcal{C}_a has Lebesgue measure zero. Each family, \mathcal{C}_a , contains both countable and uncountable sets, hence it is natural to ask about the dimensions of the sets in a given family.

This problem was first studied by Besicovitch and Taylor [3] for the case of the Hausdorff dimension. They proved that the set of attained values for the Hausdorff dimension of sets in \mathcal{C}_a was the closed interval $[0, \dim_H C_a]$, where C_a is the Cantor set associated to the sequence a . (For its definition see section 2.) A similar result was discovered by Zuberman and the last two authors in [11] in the case of packing measure. In contrast, it is not difficult to see that all the complementary sets in \mathcal{C}_a have the same lower and upper box dimension (see [6], Chapter 4).

Here we study this problem for the Assouad-type dimensions. One example of a countable set in \mathcal{C}_a is the decreasing set, denoted D_a , obtained by placing the gaps in order, beginning at L . Although this set has the smallest cardinality, we show that its Assouad dimension is maximal among all Assouad dimensions of sets in \mathcal{C}_a . Further, we prove that there are only two possible values for $\dim_A D_a$, namely 0, 1, and we characterize when these choices occur in terms of properties of the sequence. The minimum Assouad dimension is shown to be attained by the associated Cantor set C_a . The proofs of these results are given in Subsections 3.1 and 3.2.

2010 *Mathematics Subject Classification.* 28A78, 28A80.

Key words and phrases. Assouad dimension, complementary sets, Cantor sets.

The work of I. Garcia was partially supported by a grant from the Simons Foundation. The work of K. Hare was supported by NSERC 2011-44597. The work of F. Mendivil was supported by NSERC 238549.

Having found the extreme values, we also consider the structure of the set of attainable values. We give an example of a sequence a for which each complementary set has either Assouad dimension 0 or 1, and both values occur. In this example the sequence decreases very rapidly. Under the assumption of a suitably controlled rate of decay, we prove that given any $s \in [\dim_A C_a, \dim_A D_a]$ there is some $E \in \mathcal{C}_a$ such that $\dim_A E = s$. These arguments can be found in Subsection 3.3.

In Section 4, we see that in the case of the Lower Assouad dimension the opposite situation occurs: the Lower Assouad dimension is minimized by D_a and maximized by C_a . Under a suitable technical condition on the relative sizes of the a_j , we again prove that the set of attainable values of the dimension is a closed interval.

We begin, in Section 2, by introducing our terminology and notation. There we also derive formulas for the (Lower) Assouad dimensions of the associated sets C_a , generalizing the formulas found in [15] and [20] for the special case of central Cantor sets. These formulas will be very useful for the proofs given later in the paper.

2. PRELIMINARIES

2.1. Definitions and Terminology.

Assouad-type dimensions. We begin by recalling the definitions of the Assouad-type dimensions. Given a non-empty set $F \subset \mathbb{R}$ we denote by $N_r(F)$ the minimum number of closed balls of radius r needed to cover F . By $B(x, R)$ we mean the closed ball of radius R centred at x .

The *Assouad dimension* of F is defined as

$$\dim_A F = \inf \left\{ \alpha : \text{there are constants } c, \rho > 0 \text{ such that} \right. \\ \left. \sup_{x \in F} N_r(B(x, R) \cap F) \leq c \left(\frac{R}{r} \right)^\alpha \text{ for all } 0 < r < R < \rho \right\}.$$

Dually, the *Lower Assouad dimension* of F is defined as

$$\dim_L F = \sup \left\{ \alpha : \text{there are constants } c, \rho > 0 \text{ such that} \right. \\ \left. \inf_{x \in F} N_r(B(x, R) \cap F) \geq c \left(\frac{R}{r} \right)^\alpha \text{ for all } 0 < r < R < \rho \right\}.$$

Lower Assouad dimension has simply been called Lower dimension in the literature. We are calling it Lower Assouad dimension to emphasize the relationship with the Assouad dimension.

For a precompact set F we have

$$\dim_L F \leq \underline{\dim}_B F \leq \overline{\dim}_B F \leq \dim_A F,$$

while if F is compact then

$$\dim_L F \leq \dim_H F$$

(see [13, 14]). The Assouad-type dimensions are quite sensitive to the local structure of the set. Indeed, if F has an isolated point then $\dim_L F = 0$. If F is a self-similar set, then the Lower Assouad and Hausdorff dimensions coincide, and these coincide with the Assouad dimension if F satisfies the open set condition or if $F \subseteq \mathbb{R}$ satisfies the weak separation condition. For these facts and for further background information on these dimensions we refer to the papers of Fraser (et al) [7], [8], Luukkainen [16] and Olson [19].

Complementary sets. Let $a = \{a_j\}$ be a positive sequence with finite sum L and let $I = [0, L]$. Our interest is in the family \mathcal{C}_a which consists of the closed subsets E of I , of the form $E = I \setminus \bigcup_{j=1}^{\infty} U_j$, where $\{U_j\}$ is a disjoint family of open intervals contained in I , with the length of U_j equal to a_j for each j . The members of \mathcal{C}_a are called the *complementary sets of a* . One complementary set is the countable decreasing (from right to left) set $D_a = \{\sum_{i=k}^{\infty} a_i\}_k$.

We remark that if b is any rearrangement of a , then the complementary sets are the same; in particular, this is true if b is the decreasing rearrangement.

Associated Cantor sets. Another set in each class \mathcal{C}_a is what we call the *associated Cantor set* and denote by C_a . It is constructed as follows: In the first step, we remove from I an open interval of length a_1 , resulting in two closed intervals I_1^1 and I_2^1 . Having constructed the k -th step, we obtain the closed intervals $I_1^k, \dots, I_{2^k}^k$ contained in I . The next step consists in removing from each I_j^k an open interval of length a_{2^k+j-1} , obtaining the closed intervals I_{2j-1}^{k+1} and I_{2j}^{k+1} . We define $C_a := \bigcap_{k \geq 1} \bigcup_{j=1}^{2^k} I_j^k$. This construction uniquely determines an uncountable set that is homeomorphic to the classical middle-third Cantor set. Indeed, the classical Cantor set is the Cantor set associated with the sequence $\{a_i\}$ where $a_i = 3^{-n}$ if $2^{n-1} \leq i \leq 2^n - 1$. We call $a = \{a_j\}$ the *gap sequence* of C_a .

In the case when the sequence a is decreasing (i.e., $a_j \geq a_{j+1}$ for all j), we will call C_a a *decreasing Cantor set*. The classical Cantor set is such an example.

Central Cantor sets. Given a sequence of ratios $\{r_j\}$ with $0 < r_j < 1/2$, we construct a *central Cantor set* $C\{r_j\}$ in the same fashion as the classical Cantor set, but with the 2^k intervals of step k having lengths $r_1 \cdots r_k$. This set is the Cantor set, C_a , associated to the gap sequence $a = \{a_i\}$ where $a_1 = 1 - 2r_1$ and $a_i = r_1 \cdots r_n(1 - 2r_{n+1})$ if $2^n \leq i \leq 2^{n+1} - 1$. Note that this gap sequence is not necessarily decreasing, but rather lists the gaps ordered by level and repeated according to their multiplicity.

For such a choice of gap sequence a , we always have $\dim_A D_a = 1$. This is because for each k there are 2^k gaps all of the same length, say, g_k . If we choose $R = 2^k g_k$ and $r = g_k$, then it is easy to see that for a suitable point x , $N_r(B(x, R) \cap D_a) = 2^k = R/r$.

2.2. Corresponding complementary sets. Given two summable sequences, $a = \{a_n\}$ and $b = \{b_n\}$, one can assign a natural bijection between elements in \mathcal{C}_a and \mathcal{C}_b as follows (see for example [4], Sections 2 and 3 for the case of Cantor sets). Let $E \in \mathcal{C}_a$ and denote by $g_n^{(a)}$ the complementary gap of E of length a_n . For $x \in E$, let $L_a(x) = L_a(x, E) = \{n \in \mathbb{N} : g_n^{(a)} \subset [0, x]\}$, so that

$$x = \sum_{n \in L_a(x)} a_n.$$

We define

$$\pi(x) = \sum_{n \in L_a(x)} b_n.$$

Notice that $\pi : E \rightarrow \pi(E) \subset [0, \sum b_i]$ is an increasing homeomorphism and that $\pi(E) \in \mathcal{C}_b$; the last part of the observation follows by letting $g_n^{(a)} = (x, y)$, so that $(\pi(x), \pi(y))$ is the complementary gap in F of length b_n . Moreover, for any $F \in \mathcal{C}_b$ we can do the inverse process to obtain $L_b(z, F) = L_a(x, E)$, where $x = \psi(z)$, $E = \psi(F)$ and ψ the corresponding homeomorphism. By the above considerations, the map $\Pi : \mathcal{C}_a \rightarrow \mathcal{C}_b$ given by $\Pi(E) = \pi(E)$ is a bijection. We say that $E \in \mathcal{C}_a$ and $F \in \mathcal{C}_b$ are *corresponding complementary sets* if $\Pi(E) = F$.

Two sequences $\{a_n\}$ and $\{b_n\}$ are said to be *equivalent* if $a_n \sim b_n$, that is, if there is $c \geq 1$ such that

$$c^{-1} \leq \frac{a_n}{b_n} \leq c \quad \text{for all } n.$$

Recall that Assouad-type dimensions are bi-Lipschitz invariant; see [8].

Proposition 1. *If a and b are equivalent sequences, then the corresponding complementary sets in \mathcal{C}_a and \mathcal{C}_b are bi-Lipschitz equivalent and hence have the same (Lower) Assouad dimension.*

Proof. If $x < y$, where $x, y \in E$ and $E \in \mathcal{C}_a$, then by definition of π and the equivalence hypothesis,

$$\pi(y) - \pi(x) = \sum_{n \in L_a(y) \setminus L_a(x)} b_n \sim \sum_{n \in L_a(y) \setminus L_a(x)} a_n = y - x.$$

This proves the statement. \square

We say a decreasing sequence $\{a_n\}$ is *doubling* if there is a constant τ such that $a_n \leq \tau a_{2n}$ for all n . If a is a decreasing, doubling sequence with $\sum_j a_j = 1$, then it is easy to see that a is equivalent to the sequence of gaps of a central Cantor set with ratios r_k given by $1 - 2r_1 = a_1$ and

$$(1) \quad r_1 \cdots r_n (1 - 2r_{n+1}) = 2^{-n} (a_{2^n} + \cdots + a_{2^{n+1}-1})$$

for $n \geq 1$. Consequently, the Assouad-type dimensions of C_a coincide with those of this central Cantor set.

2.3. Assouad type dimensions of C_a . Given a decreasing, summable sequence $a = \{a_j\}$, we put

$$s_n^{(a)} = 2^{-n} \sum_{j=2^n}^{\infty} a_j, \text{ for } n \geq 0.$$

When the sequence a is clear we simply write s_n . This is the average length of the intervals that arise at step n in the construction of the associated decreasing Cantor set. In the special case that a is the sequence of gaps of a central Cantor set $C\{r_j\}$, then s_n is the length of the step n Cantor intervals, $r_1 \cdots r_n$. More generally, if I_j^n is any interval arising in step n of the decreasing Cantor set C_a , then $s_{n+1} \leq |I_j^n| \leq s_{n-1}$.

Conversely, given $a = \{a_j\}$ with $\sum a_j = 1$, if we define r_n by (1), then

$$r_1 \cdots r_n (1 - 2r_{n+1}) = s_n^{(a)} - 2s_{n+1}^{(a)}$$

and an easy induction argument shows that $s_n^{(a)} = r_1 \cdots r_n$.

We can calculate the Assouad-type dimensions for the sets C_a in terms of the sequence $\{s_n^{(a)}\}$.

Theorem 2. *Let a be a decreasing, summable sequence and C_a the associated Cantor set. Then*

$$(2) \quad \dim_A C_a = \inf : \left\{ \beta : \exists k_\beta, n_\beta \text{ such that } \forall k \geq k_\beta, n \geq n_\beta \right. \\ \left. \frac{n \log 2}{\log s_k / s_{k+n}} \leq \beta \right\}.$$

and

$$(3) \quad \dim_L C_a = \sup : \left\{ \beta : \exists k_\beta, n_\beta \text{ such that } \forall k \geq k_\beta, n \geq n_\beta \right. \\ \left. \frac{n \log 2}{\log s_k / s_{k+n}} \geq \beta \right\}.$$

Proof. Assouad Dimension of C_a : Let $d = \dim_A C_a$ and α equal the right hand side in (2). Given $\epsilon > 0$ there are constants c and ρ (depending on ϵ) such that if $0 < r < R < \rho$, then

$$N_r(B(x, R) \cap C_a) \leq c \left(\frac{R}{r} \right)^{d+\epsilon}.$$

Pick $k \geq k_0$ so large that $s_k < \rho$, and let I_j^{k+1} be any interval of step $k+1$ in the construction. Put $R = |I_j^{k+1}|$ and $r = s_{k+n}$ so

$$\frac{R}{r} = \frac{|I_j^k|}{s_{k+n}} \leq \frac{s_k}{s_{k+n}}.$$

Let x denote the left endpoint of interval I_j^{k+1} . As the intervals in the construction at level $k+n-1$ have length at least r and $B(x, R) \supseteq I_j^{k+1}$, it follows that $N_r(B(x, R)) \geq 2^{n-2}$. Thus

$$2^{n-2} \leq c \left(\frac{s_k}{s_{k+n}} \right)^{d+\epsilon} \text{ for all } k \geq k_0 \text{ and } n \geq 2.$$

But $\log s_k/s_{k+n} \geq n \log 2$, so

$$\begin{aligned} d + \epsilon &\geq \frac{n \log 2}{\log s_k/s_{k+n}} - \frac{\log 4c}{\log s_k/s_{k+n}} \\ &\geq \frac{n \log 2}{\log s_k/s_{k+n}} - \frac{\log 4c}{n \log 2}. \end{aligned}$$

If we choose $n_0 > 1$ such that $\log 4c/(n \log 2) < \epsilon$ for all $n \geq n_0$ then,

$$d + 2\epsilon \geq \frac{n \log 2}{\log s_k/s_{k+n}} \text{ for all } k \geq k_0, n \geq n_0.$$

It follows that $\alpha \leq d$.

To see that $d \leq \alpha$ we will show that for any $\epsilon > 0$ there are c, ρ such that if $0 < r < R < \rho$, then

$$(4) \quad N_r(B(x, R) \cap C_a) \leq c \left(\frac{R}{r} \right)^{\alpha + \epsilon}.$$

Choose k_ϵ, n_ϵ as in the definition of α , so

$$\frac{n \log 2}{\log s_k/s_{k+n}} \leq \alpha + \epsilon \text{ for all } k \geq k_\epsilon, n \geq n_\epsilon.$$

Put $\rho = s_{k_\epsilon}/2$. Consider any $0 < r < R < \rho$ and $B(x, R)$ with $x \in C_a$. Choose k, n such that

$$s_k \leq 2R < s_{k-1} \quad (\text{so } k \geq k_\epsilon),$$

and

$$s_{n+1} \leq r < s_n \quad (\text{so } n \geq k - 1).$$

If $B(x, R)$ intersects (non-trivially) at least five intervals of step $k - 1$ in C_a , then it will fully contain at least one of level $k - 2$. But this is impossible since $\text{diam } B(x, R) < s_{k-1}$ and this is less than the length of any interval of step $k - 2$. Thus $B(x, R)$ intersects at most four step $k - 1$ intervals. The (closed) balls of radius r centred at the left endpoints of the step $n + 2$ intervals contained in these step $k - 1$ intervals cover $B(x, R) \cap C_a$ and hence $N_r(B(x, R)) \leq 4 \cdot 2^{n-k+3}$. Furthermore,

$$\frac{R}{r} \geq \frac{s_k/2}{s_n}.$$

If $n = k + m$ for $m \geq 1$ and $m \geq n_\epsilon$, then $2^m \leq (s_k/s_{k+m})^{\alpha + \epsilon}$, so

$$N_r(B(x, R) \cap C_a) \leq 32 \cdot 2^m \leq 32 \left(\frac{s_k}{s_{k+m}} \right)^{\alpha + \epsilon} \leq c \left(\frac{R}{r} \right)^{\alpha + \epsilon}$$

for a suitable constant c .

Since $N_r(B(x, R) \cap C_a) \leq 32 \cdot 2^{n_\epsilon}$ if $m \leq n_\epsilon$ and is at most 32 if $n = k - 1$ or k , and $R/r \geq 1$, it follows that by making a larger choice of constant, if necessary, inequality (4) will hold for $1 \leq m \leq n_\epsilon$ and for $n = k - 1, k$. Since (4) is satisfied, $d \leq \alpha$ and hence we have $\dim_A C_a = \alpha$.

Lower Assouad Dimension of C_a : In the case when $\inf s_{k+1}/s_k > 0$, the argument that the Lower Assouad dimension of C_a is equal to the right side of (3) is similar to the argument for the Assouad dimension. The extra assumption is used to find an upper bound on R/r when $1 \leq m \leq n_\epsilon$ or $n = k - 1, k$ in the final step of the proof.

So suppose $\inf s_{k+1}/s_k = 0$. We will show that both $\dim_L C_a = 0$ and the RHS of (3) = 0 and hence the two are equal.

It is easy to see that $\dim_L C_a = 0$ if $\inf s_{k+1}/s_k = 0$: just take $R = s_k/2$ and $r = s_{k+1}/2$. Then any ball of radius R intersects at most four intervals of step k in C_a , for otherwise, it would fully contain a step $k - 1$ interval and those have length at least $s_k \geq \text{diam } B(x, R)$. Consequently, $B(x, R)$ can be covered by at most 16 balls of radius r , namely balls centred at the endpoints of the step $k + 2$ intervals contained in four step k intervals. As we can make such choices R, r with $R/r \rightarrow \infty$ it follows that we cannot satisfy $N_r(B(x, R) \cap C_a) \geq c(R/r)^\epsilon$ for any $\epsilon > 0$.

Now assume the RHS of (3) is strictly positive. Then there will be some $\gamma > 0$ such that

$$\left(\frac{s_{k+n}}{s_k}\right)^{1/n} \geq 2^{-\gamma}$$

for all k, n sufficiently large, say $k \geq K$ and $n \geq N$. In particular, for all $k \geq K$

$$2^{-\gamma/N} \leq \frac{s_{k+N}}{s_k} \leq \frac{s_{k+1}}{s_k}.$$

As N is fixed, it follows that $\inf s_{k+1}/s_k > 0$, a contradiction. \square

Corollary 3. (i) $\dim_L C_a = 0$ if and only if $\inf s_{k+1}/s_k = 0$. In particular, if a is a doubling sequence, then $\dim_L C_a > 0$.

(ii) $\dim_L C\{r_j\} = 0$ if and only if $\inf r_j = 0$.

Proof. (i) The first statement follows from the proof of the theorem. For the second statement we note that the doubling assumption implies $\inf s_{k+1}/s_k > 0$. This is because under this assumption

$$2^{-n} \sum_{j=2^n+1}^{2^{n+1}} a_j \sim a_{2^n}$$

and thus there are positive constants c_1, c_2 so that $s_{n-1} \leq c_1(a_{2^n} + s_n)$, while $s_n \geq c_2 a_{2^n}$.

(ii) is immediate since $r_j = s_j/s_{j-1}$. \square

Remark 1. It is an easy exercise to check that we have

$$(5) \quad \dim_A C_a = \inf : \left\{ \beta : \exists k_\beta, n_\beta \text{ such that } 2^n \leq \left(\frac{s_k}{s_{k+n}}\right)^\beta \quad \forall k \geq k_\beta, n \geq n_\beta \right\}$$

$$(6) \quad = \limsup_n \left(\sup_k \frac{n \log 2}{\log(s_k/s_{k+n})} \right)$$

and

$$(7) \quad \dim_L C_a = \sup : \left\{ \beta : \exists k_\beta, n_\beta \text{ such that } 2^n \geq \left(\frac{s_k}{s_{k+n}}\right)^\beta \quad \forall k \geq k_\beta, n \geq n_\beta \right\}$$

$$(8) \quad = \liminf_n \left(\inf_k \frac{n \log 2}{\log(s_k/s_{k+n})} \right).$$

Remark 2. Similar arguments show that the same formula holds if $C\{r_j\}$ is a central Cantor set. Previously, the Assouad dimension formula (6) was obtained for central Cantor sets, using other methods, by Olson et al. in [20]; see also Li et al. [15].

3. THE ASSOUD DIMENSIONS OF COMPLEMENTARY SETS

In this section we will study the Assouad dimensions of the sets that belong to a family \mathcal{C}_a . Note that throughout the paper the constant $c > 0$ appearing in inequalities may change from one occurrence to another.

3.1. Maximal Assouad dimension. We first show that the decreasing rearrangement, D_a , gives the largest Assouad dimension. We prove that its value is either 0 or 1 and we characterize when it is 0 in terms of a lacunary type condition.

Proposition 4. Let $a = \{a_j\}$ be a decreasing, summable sequence and let $D_a \in \mathcal{C}_a$ be the decreasing rearrangement. Then $\dim_A D_a = 0$ or 1. Moreover, $\dim_A D_a = 0$ if and only if there is some $\varepsilon > 0$ such that $a_j \geq \varepsilon \sum_{i=j+1}^{\infty} a_i$ for all j .

Proof. Let $x_j = \sum_{i=j}^{\infty} a_i$, so that $D_a = \{x_j\}$. Note that $a_j = x_j - x_{j+1}$ and that $a_j \geq \varepsilon \sum_{i=j+1}^{\infty} a_i$ if and only if $a_j \geq \varepsilon x_{j+1}$ if and only if $x_j \geq (1 + \varepsilon)x_{j+1}$.

First, we will assume there is no $\varepsilon > 0$ such that $a_j \geq \varepsilon \sum_{i=j+1}^{\infty} a_i$ for all j and prove $\dim_A D_a = 1$. Under this assumption, for every positive integer N_0 there are arbitrarily large k such that $a_k/x_{k+1} < 1/N_0$. Given any such k , choose $N \geq N_0$, depending on k , such that

$$(9) \quad \frac{1}{N+1} \leq \frac{a_k}{x_{k+1}} < \frac{1}{N}.$$

Let $R = x_{k+1}$ and $r = a_k$. Note that (9) implies

$$\frac{R}{r} \leq N+1 \text{ and } \frac{x_{k+1}}{N+1} \leq r < \frac{x_{k+1}}{N} < R.$$

Consider $B(0, R) \cap D_a = \{x_j : j \geq k+1\}$. As $x_{k+1} > Nr$, the intervals

$$[x_{k+1} - (s+1)r, x_{k+1} - sr] \quad \text{for } s = 0, \dots, N-1$$

are contained in $[0, x_{k+1}]$. Each has length r and as $r \geq a_i$ for $i \geq k$ by the decreasing assumption, each of these intervals must contain some x_j . Consequently,

$$N_{r/2}(B(0, R) \cap D_a) \geq \left\lfloor \frac{N}{2} \right\rfloor.$$

Since we can choose N arbitrarily large and R arbitrarily small, the fact that $R/r \leq N+1$ forces $\dim_A D_a = 1$.

Next, assume there is some $\varepsilon > 0$ such that $a_j \geq \varepsilon \sum_{i=j+1}^{\infty} a_i = \varepsilon x_{j+1}$ for all j . Putting $\lambda = 1 + \varepsilon$ we have $x_k/x_{k+1} \geq \lambda$ for all k . Choose any $0 < r < R$ and consider $B(x, R)$ for $x \in D_a$.

Case 1: $x \leq R$. Then $B(x, R) \cap D_a = [0, x+R] \cap D_a$. Choose the minimal index k such that $x_k \leq x+R$.

a) Assume there is an $i \geq k+1$ such that $a_i < r \leq a_{i-1}$. Then $r > \varepsilon x_{i+1}$ so $[0, x_{i+1}] \cap D_a$ is covered by $1/\varepsilon$ balls of radius r . On the other hand, $[x_i, x+R] \cap D_a = \{x_j\}_{j=k}^i$, hence

$$N_r(B(x, R) \cap D_a) \leq i - k + 1 + 1/\varepsilon.$$

Since $[0, x_k] \subset B(x, R)$, $R \geq x_k/2$ and by the lacunarity assumption $x_k \geq \lambda^{i-k-1}x_{i-1}$. Furthermore, $r \leq x_{i-1}$, so $R/r \geq \lambda^{i-k-1}/2$. Since $\lambda > 1$, it follows that given any $\alpha > 0$ there is a constant c depending on α such that

$$N_r(B(x, R) \cap D_a) \leq c \left(\frac{\lambda^{i-k-1}}{2} \right)^\alpha \leq c \left(\frac{R}{r} \right)^\alpha.$$

b) Otherwise, $r \geq a_k \geq \varepsilon x_{k+1}$, so $N_r(B(x, R) \cap D_a) \leq 1 + 1/\varepsilon$. Since $R/r \geq 1$, we clearly have $N_r(B(x, R) \cap D_a) \leq c(R/r)^\alpha$ for a suitable constant c .

Case 2: $x > R$. Under this assumption there are indices $K \leq J$ such that $B(x, R) = \{x_i\}_{i=K}^J$ and thus $2R \geq x_K - x_J \geq (\lambda^{J-K} - 1)x_J$. It follows easily from these observations that if $a_K \leq r$ or $r \leq a_J$, then for any $\alpha > 0$ there is a constant c depending on α such that $N_r(B(x, R) \cap D_a) \leq c(R/r)^\alpha$.

Thus we can assume there is some i with $K \leq i-1 \leq J$ such that $\varepsilon x_{i+1} \leq a_i < r \leq a_{i-1} \leq x_{i-1}$. As before, $N_r(B(x, R) \cap D_a) \leq i - K + 1 + 1/\varepsilon$. Now $x_K \geq x_{i-1} \geq x_J$, thus the lacunarity property implies

$$2R \geq x_K - x_J \geq x_K - x_{i-1} \geq (\lambda^{i-K-1} - 1)x_{i-1}.$$

Hence

$$\frac{R}{r} \geq \frac{1}{2}(\lambda^{i-K-1} - 1)$$

and, as in the first case, it follows that given any $\alpha > 0$, $N_r(B(x, R) \cap D_a) \leq c(R/r)^\alpha$ for a suitable constant c .

Since we can obtain this bound for arbitrary $\alpha > 0$, we deduce that $\dim_A D_a = 0$. \square

Proposition 5. *Suppose $a = \{a_k\}$ is a decreasing, summable sequence and that $\dim_A D_a = 0$. If $E \in \mathcal{C}_a$, then $\dim_A E = 0$.*

Proof. As $\dim_A D_a = 0$, the previous proposition implies there is some $\varepsilon > 0$ such that $a_j \geq \varepsilon \sum_{i=j+1}^{\infty} a_i$ for all j . Choose any $0 < r < R \leq a_1$ and let $x \in E$.

Pick k such that $a_k \leq r < a_{k-1}$. Let n be the total number of gaps of length a_l for some $l < k$ that are contained in $B(x, R)$. The length of the interval between any two such consecutive gaps is at most

$$a_k + \sum_{i=k+1}^{\infty} a_i \leq \left(1 + \frac{1}{\varepsilon}\right) a_k \leq r \left(1 + \frac{1}{\varepsilon}\right).$$

It follows that $N_r(B(x, R) \cap E) \leq n(1 + 1/\varepsilon)$. If the n gaps of length a_l , $l < k$, that are contained in $B(x, R)$ have lengths a_{l_1}, \dots, a_{l_n} where $l_j < l_{j+1}$, then the lacunarity assumption implies

$$\begin{aligned} R &\geq a_{l_1} + \sum_{j=2}^n a_{l_j} \geq \varepsilon \sum_{j=l_1+1}^{\infty} a_j + \sum_{j=2}^n a_{l_j} \\ &\geq (1 + \varepsilon) \sum_{j=2}^n a_{l_j} \geq (1 + \varepsilon)^{n-1} a_{l_n} \geq (1 + \varepsilon)^{n-1} a_{k-1}. \end{aligned}$$

Hence $R/r \geq (1 + \varepsilon)^{n-1}$ for all n . This proves that $\dim_A E = 0$. \square

Corollary 6. *If a is any summable, decreasing sequence and $E \in \mathcal{C}_a$ is any complementary set, then $\dim_A E \leq \dim_A D_a$.*

Proof. If $\dim_A D_a = 1$ it is clearly maximal. Otherwise it is 0 and then all other complementary sets also have Assouad dimension 0 by the Proposition. \square

Corollary 7. *Suppose $\{a_j\}$ is a positive, decreasing sequence with the property that there are positive constants c_1, c_2 and $\lambda < 1$ such that $c_1 \lambda^j \leq a_j \leq c_2 \lambda^j$. If $E \in \mathcal{C}_a$, then $\dim_A E = 0$.*

Proof. By Proposition 4, $\dim_A D_a = 0$. Now appeal to the previous Corollary. \square

3.2. Minimal Assouad dimension. Next, we prove that the decreasing Cantor set has the minimal Assouad dimension from the family \mathcal{C}_a .

Theorem 8. *If a is any summable, decreasing sequence, then $\dim_A E \geq \dim_A C_a$ for any complementary set $E \in \mathcal{C}_a$.*

Proof. We can assume $\dim_A C_a > 0$, else there is nothing to prove. The proof is divided into two cases depending on whether or not there are frequently gaps that are not ‘too small’.

First assume there is some $\delta > 0$ and a constant N_0 such that there is no string of N_0 consecutive ratios, s_j/s_{j-1} , all exceeding $1/2 - \delta$. Let $\gamma = \dim_A C_a$, fix $0 < \epsilon < \gamma/3$ and suppose $E \in \mathcal{C}_a$ has $\dim_A E = \alpha < \gamma - 3\epsilon$. Indeed, assume

$$(10) \quad N_r(B(x, R) \cap E) \leq c_E \left(\frac{R}{r}\right)^{\alpha+\epsilon} \quad \forall 0 < r < R < \rho.$$

Pick k_0 so large that $2^k s_k < \rho$ if $k \geq k_0$. By (5) we can pick k and n arbitrarily large such that

$$2^n \geq \left(\frac{s_k}{s_{k+n}}\right)^{\gamma-\epsilon} \geq 2^{n\epsilon} \left(\frac{s_k}{s_{k+n}}\right)^{\gamma-2\epsilon}.$$

By hypothesis, there is an index j with $k+n-N_0 < j \leq k+n$ and $s_{j+1}/s_j < 1/2 - \delta$. This ensures that

$$(11) \quad 2\delta s_j \leq s_j - 2s_{j+1} = 2^{-j}(a_{2^j} + \dots + a_{2^{j+1}-1}) \leq a_{2^j}$$

and therefore

$$(12) \quad 2^n \geq 2^{n\epsilon} \left(\frac{s_k}{s_j} \right)^{\gamma-2\epsilon} \geq c 2^{n\epsilon} \left(\frac{s_k}{a_{2^j}} \right)^{\gamma-2\epsilon}$$

for a suitable constant c .

Removing from $[0, L]$ the complementary gaps of E that have lengths a_1, \dots, a_{2^k-1} (i.e., the gaps of levels $1, \dots, k$), we obtain the set $J_1 \cup \dots \cup J_{M_k} \cup \{\text{singletons}\}$, where J_i are non-trivial closed intervals and $M_k < 2^k$. Note that

$$\sum_{i=1}^{M_k} |J_i| = 2^k s_k$$

as this is the sum of the remaining gaps to place.

Let b_i be the number of gaps of level $j+1$ contained in J_i . Thus if x is an endpoint of J_i and $r = a_{2^j}/2$ we have

$$N_r(B(x, |J_i|) \cap E) \geq b_i.$$

Moreover, $\sum b_i = 2^j$, so combining (12) with Hölder's inequality gives

$$\begin{aligned} \sum b_i &\geq c 2^{j-n} 2^{n\epsilon} \left(\frac{s_k}{a_{2^j}} \right)^{\gamma-2\epsilon} \\ &\geq c 2^{k-N_0} 2^{n\epsilon} \left(\frac{2^{-k} \sum_{i=1}^{M_k} |J_i|}{r} \right)^{\gamma-2\epsilon} \\ &\geq c 2^{n\epsilon} \left(\frac{2^k}{M_k} \right)^{(1-(\gamma-2\epsilon))} \frac{\sum |J_i|^{\gamma-2\epsilon}}{r^{\gamma-2\epsilon}} \\ &\geq c 2^{n\epsilon} \frac{\sum |J_i|^{\gamma-2\epsilon}}{r^{\gamma-2\epsilon}}. \end{aligned}$$

Therefore, there exists i such that

$$b_i \geq c 2^{n\epsilon} \left(\frac{|J_i|}{r} \right)^{\gamma-2\epsilon},$$

and hence

$$N_r(B(x, |J_i|) \cap E) \geq c 2^{n\epsilon} \left(\frac{|J_i|}{r} \right)^{\gamma-2\epsilon}.$$

As $|J_i| > 0$ we have $b_i \geq 1$ so J_i contains some gap of level $j+1$. This guarantees that $r \leq |J_i|$. Furthermore, $|J_i| \leq 2^k s_k < \rho$. As k and n can be made arbitrarily large that contradicts (10).

Now assume that for any $\delta > 0$ and integer N , there is a string of at least N consecutive ratios $s_j/s_{j-1} > 1/2 - \delta = \lambda$. In this case, we have $s_k/s_{k+N} < \lambda^{-N}$ for some k and hence

$$\sup_k \left(\frac{N \log 2}{\log s_k/s_{k+N}} \right) \geq \frac{\log 2}{|\log \lambda|}.$$

As we can do this for all N , it follows from (6) that $\dim_A C_a \geq \log 2 / |\log \lambda|$. Since λ can be chosen arbitrarily close to $1/2$, we see that $\dim_A C_a = 1$. Thus we need to show that any complementary set E also has Assouad dimension one.

Note that if $\overline{\dim}_B C_a = 1$, then $\overline{\dim}_B E = 1$ because of the invariance of the upper box dimension for complementary sets. Hence also $\dim_A E = 1$ and we are done.

Otherwise,

$$(13) \quad \overline{\dim}_B C_a = \limsup_{n \rightarrow \infty} \frac{n \log 2}{|\log s_n|} < 1.$$

(see [10]), hence there must exist some $\lambda_0 < 1/2$ such that $s_{j+1}/s_j \leq \lambda_0$ for infinitely many j 's. Fix $\epsilon < 1/2$ and pick $\lambda_0 < \lambda < 1/2$ such that $\lambda^{-(1-\epsilon)} < 2$. By our hypothesis and this observation, we

can take N large and an appropriate k such that all the ratios $s_{k+1}/s_k, \dots, s_{k+N}/s_{k+N-1}$ are greater than λ , but $s_{k+N+1}/s_{k+N} \leq \lambda$. Using (11) gives the estimate

$$\begin{aligned} 2^N &\geq \lambda^{-N(1-\epsilon)} \geq \left(\frac{s_k}{s_{k+N}}\right)^{1-\epsilon} \geq 2^{N\epsilon} \left(\frac{s_k}{s_{k+N}}\right)^{1-2\epsilon} \\ &\geq c 2^{N\epsilon} \left(\frac{s_k}{a_{2^{k+N}}}\right)^{1-2\epsilon} \end{aligned}$$

where $c > 0$ depends only on λ and ϵ . At this point we proceed exactly as in the previous case, but letting b_i denote the number of gaps of level $k + N + 1$. We omit the details. \square

3.3. The attainable values for the Assouad dimension. In this subsection, our interest is in determining the possible values for the $\dim_A E$ when $E \in \mathcal{C}_a$. We start with the following example that shows that in general this set need not be an interval. In fact, in this example, only two values are attained.

Example 9. Assume $\{g_k\}_{k=0}^\infty$ is a strictly decreasing, summable sequence satisfying

- (1) $\sum_{j \geq k+1} 2^j g_j < g_k/2$ (in particular, $2^k g_k \rightarrow 0$);
- (2) $(g_{k-1}/g_k)^{1/k} \geq 2^{k+4}$.

Let a be the decreasing sequence where each g_j occurs 2^j times. Then each subset in \mathcal{C}_a has either Assouad dimension 0 or 1.

Proof. Fix $E \in \mathcal{C}_a$. For each $k \geq 1$, let m_k be the maximum number of complementary intervals of length g_k between two consecutive gaps of length strictly greater than g_k ; here we consider also as gaps the unbounded intervals $(-\infty, 0)$ and (L, ∞) .

Claim 1. Suppose $\dim_A E < 1$. Then $\{m_k\}$ is bounded.

Proof of the Claim. Let $\dim_A E = \beta < \alpha < 1$ and choose c, ρ such that

$$N_r(B(x, R) \cap E) \leq c \left(\frac{R}{r}\right)^\alpha \quad \forall x \in E, \quad 0 < r < R \leq \rho.$$

Choose k_0 such that $\rho \geq 2^{k_0} g_{k_0}$ and let $k \geq k_0$.

Consider a subinterval bounded by two consecutive gaps of length strictly greater than g_k , where m_k occurs. Let x be an endpoint of the subinterval, i.e., x is an endpoint of one of the bounding gaps of size greater than g_k . Let R be the distance between the bounding gaps and $r = g_k/2$. As there are m_k gaps of length g_k between the two bounding intervals and no gaps of size g_j , $j < k$, by the first hypothesis

$$\begin{aligned} m_k g_k &\leq R \leq m_k g_k + \sum_{i > k} 2^i g_i \\ &\leq (m_k + 1) g_k. \end{aligned}$$

Note that $N_r(B(x, R) \cap E) \geq m_k$ since at least one r -ball is needed for each left endpoint of intervals of size g_k . Hence

$$m_k \leq N_r(B(x, R) \cap E) \leq c \left(\frac{R}{r}\right)^\alpha \leq c \left(\frac{(m_k + 1) g_k}{g_k}\right)^\alpha \leq 2^\alpha c m_k^\alpha.$$

Thus $m_k^{1-\alpha} \leq 2^\alpha c$. As $\alpha < 1$ this proves the sequence $\{m_k\}$ is bounded, as claimed. \square

Next, we show that any complementary set $E \in \mathcal{C}_a$, with $\{m_k\}$ bounded, has $\dim_A E = 0$. To prove this, we will show that for each $\alpha = 1/N$, there is a constant c_N such that

$$(14) \quad N_r(B(x, R) \cap E) \leq c_N \left(\frac{R}{r}\right)^\alpha \quad \forall 0 < r < R \leq 1.$$

The choice of c_N will be clear from the proof.

Case 1: $g_N/2 \leq r < R \leq 1$. Since

$$\sum_{i>N} 2^i g_i < \frac{g_N}{2} \leq r,$$

a ball of radius r will cover the intervals lying between two consecutive gaps of length greater than g_N . Hence

$$N_r(B(x, R) \cap E) \leq 2 \# \{\text{gaps } g_j, j \leq N\} < 2^{N+2}.$$

As long as $c_N \geq 2^{N+2}$ then (14) holds in this case.

Case 2: $g_{k+1}/2 < r \leq g_k/2$, $R \geq g_{k-1}/2$ for some $k \geq N$. To bound $N_r(B(x, R) \cap E)$ it will be sufficient to take all the endpoints of gaps of size g_j , $j \leq k+1$ by hypothesis (1). Therefore

$$N_r(B(x, R) \cap E) \leq 2 \cdot 2^{k+2} + 2,$$

the last 2 to count the two endpoints on $[0, L]$. Moreover, by hypothesis (2)

$$\left(\frac{R}{r}\right)^\alpha \geq \left(\frac{g_{k-1}/2}{g_k/2}\right)^\alpha \geq \left(\frac{g_{k-1}}{g_k}\right)^{1/k} \geq 2^{k+4},$$

and that together with the previous estimate shows that $c_N \geq 1$ works.

Case 3: $r = g_k/2$, $g_k/2 < R < g_{k-1}/2$. Consider $B(x, R)$. This interval cannot completely contain any gap of size g_j , $j \leq k-1$ (although either endpoint of $B(x, R)$ could be contained in such a gap). Thus $B(x, R)$ intersects $n_k \leq m_k$ gaps of size g_k and contains no full gaps of size g_i , $i < k$. Hence

$$N_r(B(x, R) \cap E) \leq 2n_k + 2 \leq 2c_0 + 2$$

where c_0 is a bound for $\{m_k\}$. As $R/r > 1$, we just pick $c_N > 2c_0 + 2$ to get

$$N_r(B(x, R) \cap E) \leq c_N \left(\frac{R}{r}\right)^{1/N}.$$

Case 4: $g_{k+1}/2 < r < g_k/2$, $g_k/2 < R < g_{k-1}/2$. Again, $B(x, R)$ does not contain any gaps of size g_j , $j \leq k-1$. If it also does not fully contain any gaps of size g_k , then it intersects at most m_{k+1} gaps of size g_{k+1} and it follows that

$$N_r(B(x, R) \cap E) \leq 2m_{k+1} + 2 \leq 2c_0 + 2$$

and we are fine. Otherwise, it intersects at most m_k gaps of size g_k . Between any two of these (or between such a gap and the end of the interval $B(x, R)$), there are at most m_{k+1} gaps of size g_{k+1} , so

$$N_r(B(x, R) \cap E) \leq 2(m_k + 1)m_{k+1} + 2m_k + 2$$

which is again uniformly bounded.

Case 5: $g_{k+1}/2 < r < g_k/2$, $g_{k+1}/2 < R < g_k/2$. This is essentially the same as the easy part of Case 4 as $B(x, R)$ admits no gaps of size g_j , $j \leq k$.

Finally, we remark that the Cantor rearrangement has $m_k = 1$ so its Assouad dimension is 0. Moreover, $\dim_A D_a = 1$. \square

Next we show that under the doubling hypothesis on the gap sequence the set of attained values is an interval.

Theorem 10. *Let $C\{r_j\}$ be a central Cantor set whose gap sequence is doubling. Then for any $s \in [\dim_A C\{r_j\}, 1]$ there is a complementary set E of the gap sequence of $C\{r_j\}$ such that $\dim_A E = s$.*

Proof. We assume $s < 1$ for otherwise we take E to be the decreasing rearrangement of Assouad dimension 1. We will construct E in such a way that $E = \mathcal{A} \cup \mathcal{B}$, with $\dim_A \mathcal{A} = s$ and $\dim_A \mathcal{B} = \dim_A C\{r_j\}$. Because of the finite stability of the Assouad dimension (see [16]) it will follow that $\dim_A E = s$.

The set \mathcal{A} will be the union of countably many finite sets A_k . Each A_k will consist of the 2^{k+1} endpoints of the k 'th step intervals in a rescaled approximation of a central Cantor set of fixed ratio $\gamma < 1/2$, where $\gamma^s = 1/2$. (This choice is made so that the central Cantor set with ratio γ has

Assouad dimension s). We will position the sets so that $\max A_{k+1} = \min A_k$, with 0 being the unique accumulation point of \mathcal{A} .

Let $a = \{a_j\}$ be the gap sequence of $C\{r_j\}$. We will let $b = \{b_j\}$ be the subsequence of a consisting of the gaps that remain after forming \mathcal{A} . We will show that b is equivalent to a and then call upon Proposition 1 to construct a suitable set \mathcal{B} . We leave the details to the end of the proof.

We begin by fixing notation that will be used throughout the proof. Let $g_k = r_1 \cdots r_{k-1}(1 - 2r_k)$ be the length of the gaps at step k in the construction of $C\{r_j\}$ and let $\alpha > 1$ be a doubling constant for $C\{r_j\}$, meaning,

$$\frac{1}{\alpha}g_{k+1} \leq g_k \leq \alpha g_{k+1} \text{ for all } k.$$

The sets A_k will be constructed by an iterative process. To start the process, we let $d_1 = g_5$ and $n_1 + 1 = \max\{i : g_i \geq d_1\}$. Then $n_1 > 3$ and

$$g_{n_1+2} < d_1 \leq g_{n_1+1}.$$

Having chosen d_j and $n_j > j + 2$ for $j = 1, \dots, k - 1$, we now let d_k be the size of a gap at level $i > k + 2$, sufficiently small so that

$$(15) \quad 2^k \leq \left(\frac{d_{k-1}}{d_k}\right)^s$$

and

$$(16) \quad \frac{\alpha^2}{1 - 2\gamma} \sum_{i=j+1}^k d_i < d_j \gamma^j \text{ for } j = 1, \dots, k - 1.$$

Put $n_k + 1 = \max\{i : g_i \geq d_k\}$. This choice ensures that $n_k > k + 2$, the sequence $\{n_k\}$ is increasing and

$$g_{n_k+2} < d_k \leq g_{n_k+1} \text{ for all } k.$$

Construction of A_k . Having chosen d_k and n_k we now proceed with the construction of A_k . We note that (16) implies

$$(17) \quad \frac{\alpha}{1 - 2\gamma} \sum_{i=1}^{\infty} d_{j+i} < d_j \gamma^j \text{ for all } j$$

and

$$(18) \quad \frac{\alpha^2}{1 - 2\gamma} d_{k+1} < d_k \gamma^k.$$

Step 1. For each $0 \leq j \leq k$, we will choose 2^j gaps from $\{a_i\}$ of lengths comparable to $d_k \gamma^j$ in length and call these the *gaps of the j -th level in A_k* . To make such a selection, we choose positive integers i_j , for $j = 0, \dots, k$, so that

$$n_k + i_j = \max\{l : g_l \geq d_k \gamma^j\}.$$

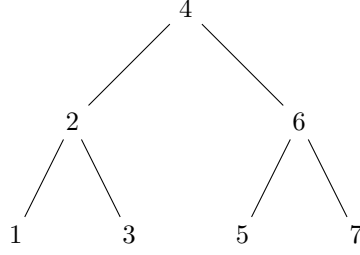
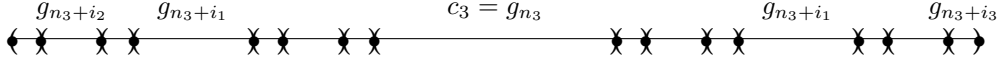
This will ensure that $i_0 = 1$, $i_j \leq i_{j+1}$,

$$(19) \quad g_{n_k+i_j+1} < d_k \gamma^j \leq g_{n_k+i_j} \text{ and } d_k \gamma^j \leq g_{n_k+i_j} \leq \alpha d_k \gamma^j,$$

the latter by the doubling condition.

We define the gaps of the j 'th level in A_k to have length $g_{n_k+i_j}$. As $n_k > k + 2$, there are sufficiently many gaps from which to make the selection, even if the i_j are not strictly increasing. In fact, we will use at most half the available gaps at any level.

Step 2. We arrange these $2^{k+1} - 1$ gaps to produce the discrete set A_k with 2^{k+1} points in the following manner. Consider the full binary tree \mathcal{T}_k up to level k . For each $j = 0, \dots, k$ place the 2^j indices of the gaps of level j in the vertices of level j of \mathcal{T}_k . Then paste the corresponding gaps from left to right using the in-order tree traversal; see Figure 1 to recall the definition of this order when $k = 2$. This

FIGURE 1. The order given by \mathcal{T}_2 .FIGURE 2. A_3 has 2^4 points; the sizes of some gaps are indicated.

will be the set A_k ; see Figure 2. Since A_k is made with 2^j gaps of level j for $0 \leq j \leq k$, by (19) we have

$$(20) \quad d_k \leq g_{n_k+1} \leq \text{diam } A_k \leq \sum_{j=0}^k 2^j g_{n_k+i_j} \leq \alpha d_k \sum_{j=0}^k 2^j \gamma^j < \frac{\alpha}{1-2\gamma} d_k.$$

We remark that conditions (18),(19) and the doubling property imply that

$$g_{n_k+i_j+2} \geq \frac{1}{\alpha} g_{n_k+i_j+1} \geq \frac{1}{\alpha^2} g_{n_k+i_j} \geq \frac{1}{\alpha^2} d_k \gamma^j > d_{k+1},$$

thus the gaps associated with sets A_k and A_m , for $k \neq m$, correspond to different gap levels in $C\{r_j\}$.

We set $\rho_k = d_k \gamma^k$ and $R_k = \text{diam } A_k$. Both sequences are decreasing, $\rho_k < R_k$ and by (17) and (20) we obtain the useful bounds

$$(21) \quad R_{k+1} \leq \text{diam} \left(\bigcup_{j=k+1}^{\infty} A_j \right) = \sum_{j=1}^{\infty} R_{k+j} \leq \frac{\alpha}{1-2\gamma} \sum_{j=1}^{\infty} d_{k+j} < d_k \gamma^k = \rho_k < R_k.$$

Having constructed the sets A_k , our next task is to prove that $\dim_{\mathcal{A}} \mathcal{A} = s$. In this part of the argument c will denote a constant that depends only upon α and γ .

By considering the special case $R = R_k$ and $r = \rho_k$, we first verify that s is a lower bound for the dimension. In fact, for any $y \in A_k$ the set $B(y, R) \cap A_k$ contains at least 2^k points spread r apart (one for each k 'th level gap of A_k), hence the choice of $\gamma = 2^{-1/s}$ and (20) yields

$$N_r(B(y, R) \cap \mathcal{A}) \geq 2^k = \gamma^{-ks},$$

while

$$\gamma^{-ks} = \left(\frac{d_k}{d_k \gamma^k} \right)^s \leq \left(\frac{R_k}{\rho_k} \right)^s \leq \left(\frac{\frac{\alpha}{1-2\gamma} d_k}{d_k \gamma^k} \right)^s = c \gamma^{-ks}.$$

Since the last quotient goes to infinity as $k \rightarrow \infty$, we have $\dim_{\mathcal{A}} \mathcal{A} \geq s$.

To prove that $\dim_{\mathcal{A}} \mathcal{A} \leq s$ we will show that there is a constant c such that for all $0 < r < R \leq \text{diam } \mathcal{A}$ and $x \in \mathcal{A}$, $N_r(B(x, R) \cap \mathcal{A}) \leq c(R/r)^s$.

Given such $r < R$, choose m and $l \leq m+1$ such that

$$\rho_m \leq r < \rho_{m-1} \quad \text{and} \quad R_l \leq R < R_{l-1}.$$

From (21) we deduce that $\bigcup_{j=m+1}^{\infty} A_j$ is contained in a ball of radius $\rho_m \leq r$, hence for all x and R ,

$$(22) \quad N_r \left(B(x, R) \cap \bigcup_{j=m+1}^{\infty} A_j \right) \leq 1.$$

Another important observation is that all gaps in A_j have length at least $d_j \gamma^j = \rho_j$, hence

$$(23) \quad N_r \left(B(x, R) \cap \bigcup_{j=1}^J A_j \right) \leq 3 \quad \text{if } R \leq \rho_J.$$

We now divide the proof that $\dim_A \mathcal{A} \leq s$ into three cases.

Case 1: $l < m$. Here we use (22) and the observation that for any set F , $N_r(F) \leq \#F$, to get the simple bound

$$\begin{aligned} N_r(B(x, R) \cap \mathcal{A}) &\leq N_r \left(B(x, R) \cap \bigcup_{j=m+1}^{\infty} A_j \right) + N_r \left(B(x, R) \cap \bigcup_{j=1}^m A_j \right) \\ &\leq 1 + \sum_{j=1}^m \#A_j < 8 \cdot 2^{m-1}. \end{aligned}$$

Furthermore,

$$\left(\frac{R}{r} \right)^s \geq \left(\frac{R_{m-1}}{\rho_{m-1}} \right)^s \geq \left(\frac{d_{m-1}}{d_{m-1} \gamma^{m-1}} \right)^s = 2^{m-1},$$

consequently, $N_r(B(x, R) \cap \mathcal{A}) \leq 8(R/r)^s$.

Case 2a): $l = m$ and $R \leq \rho_{m-1}$.

From (22) and (23) we deduce that $N_r(B(x, R) \cap \bigcup_{j \neq m} A_j) \leq 4$, so it suffices to study $N_r(B(x, R) \cap A_m)$.

If $r \geq d_m$, then $\text{diam } A_m = R_m \leq \alpha d_m / (1 - 2\gamma) \leq cr$ and hence $N_r(B(x, R) \cap \mathcal{A}) \leq c + 4$ and we are done.

Otherwise, as $r \geq \rho_m = d_m \gamma^m$, there exists some $0 \leq j < m$ such that

$$d_m \gamma^{j+1} \leq r < d_m \gamma^j,$$

and then

$$\left(\frac{R}{r} \right)^s \geq \left(\frac{R_m}{d_m \gamma^j} \right)^s \geq \left(\frac{d_m}{d_m \gamma^j} \right)^s = 2^j.$$

The set A_m is contained in the union of the disjoint closed intervals bounded by consecutive gaps of A_m of level $\leq j$. There are 2^{j+1} of these intervals and they each have length

$$(24) \quad \sum_{l=1}^{m-j} 2^{l-1} g_{n_m+i_{j+l}} \leq \sum_{l=1}^{m-j} 2^{l-1} \alpha d_m \gamma^{j+l} < \frac{\alpha}{1-2\gamma} d_m \gamma^{j+1} \leq cr.$$

Thus $N_r(B(x, R) \cap A_m) \leq c2^j$ and hence

$$N_r(B(x, R) \cap \mathcal{A}) \leq c2^j \leq c \left(\frac{R}{r} \right)^s.$$

Case 2 b): $l = m$ and $R > \rho_{m-1}$.

As (21) tells us $R_{m-1} < \rho_{m-2}$ and we are given $R < R_{m-1}$, from (23) we deduce that

$$N_r \left(B(x, R) \cap \bigcup_{i \leq m-2} A_i \right) \leq 3.$$

Thus it suffices to bound $N_r(B(x, R) \cap (A_{m-1} \cup A_m))$ by $c(R/r)^s$. Of course, $N_r(B(x, R) \cap (A_{m-1} \cup A_m))$ is dominated by the cardinality of $A_{m-1} \cup A_m$, hence if $R \geq d_{m-1}$, then we have

$$N_r(B(x, R) \cap \mathcal{A}) \leq c2^m \leq c \left(\frac{d_{m-1}}{d_{m-1} \gamma^{m-1}} \right)^s \leq c \left(\frac{R}{r} \right)^s.$$

So we can assume there is some $1 \leq h < m$ such that

$$d_{m-1}\gamma^h \leq R < d_{m-1}\gamma^{h-1}.$$

In this case, R is smaller than the size of any gap in A_{m-1} at level less than h . It follows that $B(x, R) \cap A_{m-1}$ is contained in the closed interval between two consecutive gaps of level $< h$ and therefore

$$\#(B(x, R) \cap A_{m-1}) \leq c2^{m-h}.$$

If, also, $r \geq d_m$, then $\text{diam } A_m \leq cr$, so $N_r(A_m) \leq c$. Putting these facts together we see that $N_r(B(x, R) \cap \mathcal{A}) \leq C2^{m-h}$, while (15) ensures that

$$\left(\frac{R}{r}\right)^s \geq \left(\frac{d_{m-1}\gamma^h}{d_m}\right)^s = 2^{m-h}.$$

Otherwise, there exists $1 \leq j < m$ such that

$$d_m\gamma^{j+1} \leq r < d_m\gamma^j.$$

As in case 2 a), this hypothesis ensures $N_r(B(x, R) \cap A_m) \leq c2^j$ so that

$$N_r(B(x, R) \cap \mathcal{A}) \leq c(2^{m-h} + 2^j),$$

while $(R/r)^s \geq (d_{m-1}\gamma^h/d_m\gamma^j)^s = 2^{m+j-h}$.

Case 3: $l = m + 1$. Then we have $\rho_m \leq r < R < R_m < \rho_{m-1}$.

If $d_m \leq r$, then $R \leq cr$, hence $N_r(B(x, R) \cap \mathcal{A}) \leq c$ and we are done.

So assume $d_m\gamma^j \leq r < d_m\gamma^{j-1}$ for some $1 \leq j \leq m$ and

$$d_m \leq R < R_m \quad \text{or} \quad d_m\gamma^h \leq R < d_m\gamma^{h-1} \text{ for some } 1 \leq h \leq j.$$

In either case, as $R < \rho_{m-1}$ inequalities (22) and (23) show that we need only study $N_r(B(x, R) \cap A_m)$. The size of R guarantees that $B(x, R) \cap A_m$ is contained in the interval between two consecutive gaps of the h 'th level in A_m (or simply the interval containing A_m in the first case) and is contained in the union of the closed subintervals bounded by gaps of level $\leq j$. From (24), these subintervals have length $\leq cr$ and there are at most 2^{j+1-h} (with $h = 0$ in the first case) of them. Hence

$$N_r(B(x, R) \cap A_m) \leq c2^{j-h} = c\gamma^{(h-j)s} \leq c\left(\frac{R}{r}\right)^s.$$

This completes the proof that $\dim_{\mathcal{A}} \mathcal{A} = s$.

To conclude the proof of the theorem we construct the set \mathcal{B} . First, we establish that the sequences a and b are equivalent. Recall that $a_{2^k+i} = g_k$, for $0 \leq i < 2^k$, from the fact that the Cantor set is central. The choice of our construction implies that $b_j = a_{j+t_j}$ for some $0 \leq t_j \leq j$. Thus if k is chosen with $2^k \leq j < 2^{k+1}$, then either $2^k \leq j + t_j < 2^{k+1}$, in which case $b_j = a_j = g_k$, or $2^{k+1} \leq j + t_j < 2^{k+2}$, in which case $b_j = a_{j+t_j} = g_{k+1}$. In either case, $1/\alpha \leq b_j/a_j \leq \alpha$ and so the sequences are equivalent. Now let $\mathcal{B}' \in \mathcal{C}_b$ be the complementary set corresponding to $C\{r_j\} \in \mathcal{C}_a$. Proposition 1 implies that $\dim_{\mathcal{A}} \mathcal{B}' = \dim_{\mathcal{A}} C\{r_j\}$. The set \mathcal{B} will be a translate of \mathcal{B}' that lies to the right of \mathcal{A} . Then take $E = \mathcal{A} \cup \mathcal{B} \in \mathcal{C}_a$. \square

Remark 3. We remark that this argument actually shows that given *any* $s \in (0, 1)$ there is a set \mathcal{A} whose complementary gaps are a subset of those of $C\{r_j\}$ and which has Assouad dimension s .

Corollary 11. *Suppose a is a decreasing, summable, doubling sequence. Then*

$$\{\dim_{\mathcal{A}} E : E \in \mathcal{C}_a\} = [\dim_{\mathcal{A}} C_a, 1]$$

Proof. As a is doubling, the sequence of gaps of the central Cantor set $C\{r_j\}$ with ratios r_k given by (1) is equivalent to a . In particular, $C\{r_j\}$ is doubling and $\dim_{\mathcal{A}} C\{r_j\} = \dim_{\mathcal{A}} C_a$. Thus given any $s \in [\dim_{\mathcal{A}} C_a, 1]$ the theorem implies there is a rearrangement E' of the gaps of $C\{r_j\}$ that has Assouad dimension s . The corresponding rearrangement $E \in \mathcal{C}_a$ also has dimension s by Proposition 1.

The fact that $\dim_A C_a$ is a lower bound on the set of attainable values of the Assouad dimension is the content of Theorem 8. \square

4. LOWER ASSOUD DIMENSIONS OF COMPLEMENTARY SETS

In this section we will prove that C_a has the maximal Lower Assouad dimension among all the complementary sets in \mathcal{C}_a and that under a mild technical assumption, the set of attainable Lower Assouad dimensions from the class \mathcal{C}_a is the interval $[0, \dim_L C_a]$.

Theorem 12. *If a is any summable, decreasing sequence, then $\dim_L E \leq \dim_L C_a$ for any complementary set $E \in \mathcal{C}_a$.*

We begin with a technical lemma.

Lemma 13. *Suppose $\alpha = \dim_L C_a \in (0, 1)$. Then for any $0 < \epsilon < 1 - \alpha$ there exist $\lambda < 1/2$ and arbitrarily large k and m such that $s_{k+1}/s_k \leq \lambda$ and*

$$2^m < \left(\frac{s_k}{s_{k+m}} \right)^{\alpha+\epsilon}.$$

Proof. Since $\alpha > 0$, we have $t := \inf s_{j+1}/s_j > 0$ and from formula (7) we deduce that for arbitrarily large M and K ,

$$2^M < \left(\frac{s_K}{s_{K+M}} \right)^{\alpha+\epsilon}.$$

Pick $\lambda < 1/2$ and $b < 1$ so that $\lambda^{\alpha+\epsilon} b > 1/2$. If $s_{K+1}/s_K \leq \lambda$ we are done. So assume otherwise and let $H = \max\{1 \leq i \leq M : s_{K+j}/s_{K+j-1} > \lambda \text{ for } j = 1, \dots, i\}$. Then

$$(25) \quad 2^M < \left(\frac{s_K}{s_{K+M}} \right)^{\alpha+\epsilon} < 2^H b^H \left(\frac{s_{K+H}}{s_{K+M}} \right)^{\alpha+\epsilon} \leq \frac{2^H b^H}{t^{(M-H)(\alpha+\epsilon)}},$$

so

$$(2t^{\alpha+\epsilon})^{M-H} \leq b^H.$$

This proves that not only is $H < M$, but also that $\sup_M (M - H) = \infty$. Letting $k = K + H$ and $m = M - H$, we obtain by (25) that

$$2^m = 2^{M-H} < b^H \left(\frac{s_k}{s_{k+m}} \right)^{\alpha+\epsilon}.$$

Since $s_{k+1}/s_k \leq \lambda$ by the definition of H and $b < 1$, the lemma holds. \square

Proof of Theorem 12. We assume that $\alpha = \dim_L C_a < 1$ for otherwise there is nothing to prove. Let $\epsilon > 0$ be such that $\alpha + \epsilon < 1$. If $\alpha = 0$ then, as noted in Corollary 3, $\inf s_{j+1}/s_j = 0$, so there are arbitrarily large k such that

$$(26) \quad 2 < \left(\frac{s_k}{s_{k+1}} \right)^\epsilon.$$

Otherwise, $\alpha > 0$ and by the previous lemma there is a constant $\lambda < 1/2$ such that we can always find arbitrarily large k and m such that $s_{k+1}/s_k \leq \lambda$ and

$$2^m < \left(\frac{s_k}{s_{k+m}} \right)^{\alpha+\epsilon}.$$

Put $R = 2\delta s_k$ where $\delta = 1/2 - (1/2)^{1/\epsilon}$ in the first case and $\delta = 1/2 - \lambda$ in the second; R is at most the length of any gap of level $\leq k - 1$. If we let $r = s_{k+m}$ where $m = 1$ in the first case, then

$$\left(\frac{R}{r} \right)^{\alpha+\epsilon} = \left(\frac{2\delta s_k}{s_{k+m}} \right)^{\alpha+\epsilon} > c 2^m$$

for some constant $c > 0$. Note that we can arrange for R to be arbitrarily small and R/r to be arbitrarily large.

Consider the sets $B(x_j, R) \cap E$ where x_j are the left endpoints of the gaps of level $k-1$ in E . There are $2^k - 1$ such sets and the choice of R ensures they are disjoint. Each set $B(x_j, R) \cap E$ is a subset of the union of the closed intervals which lie between consecutive gaps of levels $1, \dots, k+m$. The total length of these intervals, summed over all j , is $2^{k+m} s_{k+m}$ and there are at most 2^{k+m+1} of them. Each of these intervals, I_i , can be covered by $1 + (|I_i|/r)$ balls of radius r , hence

$$\begin{aligned} \sum_{j=1}^{2^k-1} N_r(B(x_j, R) \cap E) &\leq \sum_i \left(1 + \frac{|I_i|}{r}\right) \\ &\leq \frac{1}{r} \sum_i |I_i| + 2^{k+m+1} \leq c 2^{k+m}. \end{aligned}$$

Consequently, there is some index j such that

$$N_r(B(x_j, R) \cap E) \leq c 2^m \leq c \left(\frac{R}{r}\right)^{\alpha+\epsilon}.$$

It follows that $\dim_L E \leq \alpha$. □

Theorem 14. *Suppose $\sup r_j < 1/2$. Then given any $\alpha < \dim_L C\{r_j\}$ there is some complementary set E of the gap sequence of $C\{r_j\}$ such that $\dim_L E = \alpha$.*

Proof. If $\alpha = 0$, we simply let E be any countable rearrangement and then $\dim_L E = 0$.

So assume $\alpha > 0$. Then $\dim_L C\{r_j\} > 0$, hence $\inf r_j > 0$ and so there is some $\varepsilon > 0$ such that $\varepsilon < r_j < 1/2 - \varepsilon$ for all j .

Choose $d < 1/2$ such that $\alpha = \log 2 / |\log d|$. As $\alpha < \dim_L C\{r_j\}$, formula (8) shows that there is some N such that for all $n \geq N$ and for all k ,

$$\frac{\log 2}{|\log d|} < \frac{n \log 2}{|\log(r_{k+1} \cdots r_n)|}.$$

Taking $k = 0$, we see that $s_n = r_1 \cdots r_n > d^n$.

For each $j \geq N$, choose k_j such that

$$s_{k_j} \leq d^j < s_{k_j-1}.$$

As $r_{k_j} > \varepsilon$, we have $s_{k_j-1} \geq s_{k_j} \geq \varepsilon s_{k_j-1}$, so $d^j \sim s_{k_j}$. The sequence $\{k_j\}$ is increasing and the fact that $s_n > d^n$ implies $k_j > j$. In particular, if $k_j = \cdots = k_{j+m}$, then $k_j > j+m$ and m is bounded.

We will form a Cantor set C_b associated to a subsequence b of the gap sequence a of $C\{r_j\}$. Indeed, this subsequence will be chosen in such a way that the lengths of the 2^{j-N} gaps at step $j-N+1$ in the construction of C_b will be the lengths of the gaps of step k_j of the construction of $C\{r_j\}$. (C_b will also be a central Cantor set, but of diameter less than 1.) We note that the property that $k_j > j$ ensures that we will use at most half the available gaps from each step k_j .

The gaps of step j in $C\{r_i\}$ have length comparable to $r_1 \cdots r_j$, thus the gaps of step $j-N+1$ in C_b will be comparable in size to s_{k_j} and hence to d^j . Therefore

$$\begin{aligned} s_n^{(b)} &= \frac{1}{2^n} \sum_{k=2^n}^{\infty} b_k = \frac{1}{2^n} \sum_{i=n+1}^{\infty} 2^{i-1} \cdot \text{length of gaps of step } i \\ &\sim \frac{1}{2^n} \sum_{i=n+1}^{\infty} 2^i d^{i+N} \sim d^n. \end{aligned}$$

Consequently, $s_{k+n}^{(b)} / s_k^{(b)} \sim d^n$ and an application of formula (8) shows that $\dim_L C_b = \log 2 / |\log d| = \alpha$.

Now let b' be the sequence consisting of the remaining terms of the gap sequence a of $C\{r_j\}$. The same arguments as at the conclusion of the proof of Theorem 10 show that b' is equivalent to a . Hence, the complementary set $A \in \mathcal{C}_{b'}$ corresponding to $C\{r_j\} \in \mathcal{C}_a$ and $C\{r_j\}$ have the same Lower

Assouad dimension. Define the set E to be the union of C_b and a translation of A that lies strictly to the right of C_b . Since the Lower Assouad dimension of the union of two sets with positive separation is the minimum of the Lower Assouad dimensions of the two sets, $\dim_L E = \alpha$ (see [8]). \square

Remark. The assumption that $\sup r_j < 1/2$ is not a necessary condition for the conclusion of the Theorem. It can be shown, for example, that if $r_j = 1/2 - 1/(2j)$, then the same property holds for $C\{r_j\}$. We omit the details.

Corollary 15. *Suppose a is a decreasing, summable sequence and that $\sup s_j/s_{j-1} < 1/2$. Then*

$$\{\dim_L E : E \in \mathcal{C}_a\} = [0, \dim_L C_a].$$

Proof. Of course, $\dim_L D_a = 0$ and the fact that $\dim_L C_a$ is an upper bound on the attainable dimensions is the content of Theorem 12.

So assume $0 < \alpha < \dim_L C_a$. Then it must be the case that $\inf s_j/s_{j-1} > 0$ and it is a routine exercise to see that this fact, combined with the assumption that $\sup s_j/s_{j-1} < 1/2$, implies a is a doubling sequence. Consequently, a is equivalent to the gap sequence of a central Cantor set with ratios $\{r_j\}$ as defined in (1). Since $r_1 \cdots r_j = s_j$ it is immediate that $C\{r_j\}$ satisfies the hypothesis of the proposition. The corollary follows since Proposition 1 implies the Lower Assouad dimensions of the sets in \mathcal{C}_a have the same dimensions as the corresponding complementary sets of the gap sequence of $C\{r_j\}$. \square

We conclude with an application.

Corollary 16. *Suppose there is some $E \in \mathcal{C}_a$ such that $\dim_L E = \dim_A E = s$. Then, also, $\dim_L C_a = \dim_A C_a = \dim_H C_a = s$.*

Proof. From our theorems,

$$\dim_L E \leq \dim_L C_a \leq \dim_H C_a \leq \dim_A C_a \leq \dim_A E,$$

and thus the hypothesis implies we have equality throughout. \square

Remark. In particular, if $E \in \mathcal{C}_a$ is Ahlfors regular, then $\dim_L E = \dim_A E$ and thus $\dim_H E = \dim_H C_a$, a fact which can also be shown by other well known arguments.

Remark. It would be interesting to understand the structure of the sets $\{\dim_A E : E \in \mathcal{C}_a\}$ or $\{\dim_L E : E \in \mathcal{C}_a\}$ when a is not a doubling sequence (or does not satisfy the separation condition $\sup s_j/s_{j-1} < 1/2$ in the second case). In particular, we do not know if it is possible to obtain the full interval $[0, 1]$ in either case, or the two element set $\{0, 1\}$ in the second case.

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